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INEQUALITIES INVOLVING EXTENDED k-GAMMA AND k-BETA FUNCTIONS

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ABSTRACT. Our aim in this present paper is to introduce some inequalities such as Chebeshev's inequality, log-convexity, Hölder inequality etc. which involving the extended k-gamma and k-beta function recently introduced by Mubeen *et al.* (J. math. anal. Volume 7 Issue 5(2016), 118-131). The obtained inequalities for extended k-beta function are the generalization of inequalities of extended beta function recently proved by Mondal (J. Inequal. Appl. (2017) 2017:10). Also, these inequalities are the extended form of the some inequalities involving k-gamma and k-beta functions earlier proved by Rehman *et al.* (J. Inequal. Appl., 224(1): 2014).

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1. INTRODUCTION

In 1994, Chaudhry and Zubair [1] have introduced the following extension of gamma function

$$\Gamma_b(z) = \int_0^\infty t^{z-1} e^{-t-bt^{-1}} dt, \quad Re(z) > 0, p \geq 0. \tag{1}$$

When $b = 0$, then Γ_b tends to the classical gamma function Γ. In 1997, Chaudhry et al. [2] presented the following extension of Euler's beta function

$$B(x, y; b) = \int_0^\infty t^x (1-t)^y e^{-\frac{b}{t(1-t)}} dt \tag{2}$$

(where $Re(b) > 0, Re(x) > 0, Re(y) > 0$). When $b = 0$, then $B_0(x, y) = B(x, y)$.

In recent years, some extension of the well-known special functions have been considered by several authors (see [3]-[6]). Diaz et al. ([7]-[9]) have introduced k-gamma and k-beta functions and proved a number of their properties. They have also studied k-zeta functions and k-hypergeometric functions based on Pochhammer k-symbols for factorial functions. For $k > 0$ and $z \in \mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{z}{k}-1}}{(z)_{n,k}}.$$

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Its integral representation is also given by,

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{\frac{-t^k}{k}} dt$$

and

$$\Gamma_k(z+k) = z\Gamma_k(z)$$

The relation between Pochhammer k-symbol and k-gamma function is given as

$$(z)_{n,k} = \frac{\Gamma_k(z+nk)}{\Gamma_k(z)}.$$

The k-beta function is defined by

$$B_k(x,y) = \frac{1}{k} \int_0^\infty t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt. \tag{3}$$

The relation between k-gamma function and k-beta function is

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, Re(x) > 0, Re(y) > 0. \tag{4}$$

These studies were then followed by works of Mansour [14], Kokologiannaki [11], Krasniqi [12, 13] and Merovci [15] elaborating and strengthening the scope of k-gamma and k-beta functions.
Recently Mubeen *et al.* [16] introduced extended k-gamma and$k$-beta functions defined by:

$$\Gamma_{b,k}(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}} dt. \tag{5}$$

and

$$B_k(x,y;b) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} e^{-\frac{b^k}{kt(1-t)}} dt. \tag{6}$$

respectively. Clearly if $b = 0$, then (6) will reduces to the well known k-beta function (3).
In the same paper, they proved various properties of extended k-gamma and extended k-beta function. Also they defined further generalization of the extended k-gamma and k-beta function and k-beta distribution.

2. Inequalities involving the Extended k-beta Functions

In this section, we apply some classical integral inequalities such as Chebychev's inequality for Synchronous and asynchronous mapings, Hölder-Rogers inequality. We will prove several inequalities for extended k-beta functions. For this purpose of our study we need to recall the following well known result

Theorem 2.1. *(Chebyshev's integral inequality* [17], *p. 40) If* f, g:$[a, b] \to \mathbb{R}$ *are synchronous integrable functions and let* $h : [a, b] \to \mathbb{R}$ *be a positive integrable function, then the following result holds:*

$$\int_a^b h(t)f(t)dt \int_a^b h(t)g(t)dt \leq \int_a^b h(t)dt \int_a^b h(t)f(t)g(t)dt. \tag{7}$$

The inequality (7) is reversed if f *and* g *are asynchronous.*

Theorem 2.2. *Let* x, y, $x_1, y_1 > 0$ *such that* $(x - x_1)(y - y_1) \geq 0$, *then*

$$B_{b,k}(x, y_1)B_k(x_1, y) \leq B_{b,k}(x_1, y_1)B_{b,k}(x, y), \tag{8}$$

for all $b \geq 0$.

Proof. Consider $f(t) = t^{\frac{x-x_1}{k}}$, $g(t) = (1-t)^{\frac{y-y_1}{k}}$ and

$$h(t) = \frac{1}{k}t^{\frac{x_1}{k}-1}(1-t)^{\frac{y_1}{k}-1}\exp[-\frac{b^k}{kt(1-t)}].$$

Obviously, h is non negative on $[0, 1]$. Since $(x - x_1)(y - y_1) \geq 0$, it follows that $f'(t) = \frac{1}{k}(x - x_1)t^{\frac{x-x_1}{k}-1}$ and $g'(t) = \frac{1}{k}(y - y_1)t^{\frac{y-y_1}{k}-1}$ have the same monotonicity on $[0, 1]$ for $k > 0$.
Applying Chebeshev's integral inequality (8) for f, g and h, we have

$$\left(\frac{1}{k}\int_a^b t^{\frac{x}{k}-1}(1-t)^{\frac{y_1}{k}-1}\exp[-\frac{b^k}{kt(1-t)}]dt\right)\left(\frac{1}{k}\int_a^b t^{\frac{x_1}{k}-1}(1-t)^{\frac{y}{k}-1}\exp[-\frac{b^k}{kt(1-t)}]dt\right)$$
$$\leq \left(\frac{1}{k}\int_a^b t^{\frac{x_1}{k}-1}(1-t)^{\frac{y_1}{k}-1}\exp[-\frac{b^k}{kt(1-t)}]dt\right)\left(\frac{1}{k}\int_a^b t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\exp[-\frac{b^k}{kt(1-t)}]dt\right)$$

which is equivalent to (8). □

Corollary 2.3. *For* m, $p > 0$, *the following inequality holds for extended* k*-beta function*

$$B_{b,k}(x, x_1) \geq \Big[\beta_k(x, x)\,\beta_k(x_1, x_1)\Big]^{\frac{1}{2}},\ k > 0. \tag{26}$$

Proof. Setting $y_1 = x$ and $y = x_1$ in Theorem 2.2, we get Corollary 2.3.

$$B_{b,k}(x, x)B_{b,k}(x_1, x_1) \leq B_{b,k}(x, x_1)B_{b,k}(x_1, x) = [B_{b,k}(x, x_1)]^2.$$

□

Theorem 2.4. *Let* m, p *and* r *be positive real numbers such that* $p > r > -m$. *If* $r(p - m - r) \geq (\leq)0$, *then*

$$\Gamma_{b,k}(m)\Gamma_{b,k}(p) \geq (\leq)\Gamma_{b,k}(p - r)\Gamma_{b,k}(m + r). \tag{9}$$

Proof. Let us define the mappings $f, g, h : [0, \infty) \to [0, \infty)$ given by

$$f(t) = t^{p-r-m}, \quad g(t) = t^r, \quad h(t) = t^{m-1}e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}.$$

If $r(p - m - r) \geq (\leq)0$, then we can claim that the mappings f and g are synchronous (asynchronous) $]0, \infty[$. Thus, by using Chebychev inequality for the interval $I = (0,\infty)$ along with the functions f, g and h defined above, we can write

$$\int_0^\infty t^{m-1}e^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt\int_0^\infty t^{p-r-m}t^rt^{m-1}e^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt$$
$$\geq (\leq)\int_0^\infty t^{p-r-m}t^{m-1}e^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt\int_0^\infty t^rt^{m-1}e^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt.$$

This implies that

$$\int_0^\infty t^{m-1}e^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt\int_0^\infty t^{p-1}e^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt$$
$$\geq (\leq)\int_0^\infty t^{p-r-1}e^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt\int_0^\infty t^{m+r-1}e^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt.$$

By (5), we get the required inequality (9). □

Corollary 2.5. *If $p > 0$ and $q \in \mathbb{R}$ with $|q| < p$, then*

$$\Gamma_k(p) \leq [\Gamma_k(p-q)\Gamma_k(p+q)]^{\frac{1}{2}}. \tag{10}$$

Proof. By setting $b = 0$, $m = p$ and $r = q$ in Theorem 2.4, then we get $r(p-m-r) = -q^2 \leq 0$ and the relation (4) provides the desired Corollary 2.5. For complete study of corollary 2.5 the readers refer to [21]. □

Definition. *Two positive real numbers m and n are said to be similarly (oppositely) unitary if (see (see* [19]*)*

$$(m-1)(n-1) \geq (\leq) 0. \tag{11}$$

Theorem 2.6. *If $m, n > 0$ are similarly (oppositely) unitary, then*

$$\Gamma_{b,k}(m+n+k-1) \geq (\leq) \frac{\Gamma_{b,k}(m+k)\Gamma_{b,k}(n+k)}{\Gamma_{b,k}(k+1)}. \tag{12}$$

Proof. Consider the mappings $f, g, h : [0,\infty) \to [0,\infty)$ defined by

$$f(t) = t^{m-1}, \quad g(t) = t^{n-1}, \quad h(t) = t^k e^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}.$$

Now if the condition $(m-1)(n-1) \geq (\leq)0$ holds, then Chebychev integral inequality along with the functions f, g and h defined above is obtained as

$$\int_0^\infty t^k e^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt\int_0^\infty t^{m-1}t^{n-1}t^ke^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt$$
$$\geq (\leq)\int_0^\infty t^{m-1}t^ke^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt\int_0^\infty t^{n-1}t^ke^{-\frac{t^k}{k}-\frac{b^kt^{-k}}{k}}dt.$$

This implies that

$$\int_0^\infty t^k e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt \int_0^\infty t^{m+n+k-2} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt$$
$$\geq (\leq) \int_0^\infty t^{m+k-1} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt \int_0^\infty t^{n+k-1} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt.$$

By the definition of extended k-gamma function, we have

$$\Gamma_{b,k}(k+1)\Gamma_{b,k}(m+n+k-1) \quad \geq (\leq) \quad \Gamma_{b,k}(m+k)\Gamma_{b,k}(n+k),$$

or

$$\Gamma_{b,k}(m+n+k-1) \quad \geq (\leq) \quad \frac{\Gamma_{b,k}(m+k)\Gamma_{b,k}(n+k)}{\Gamma_{b,k}(k+1)}.$$

□

remark 2.7. *If $b=0$, then we have the results of classical k-gamma function see* [21].

Theorem 2.8. *If m and n are positive real numbers such that m and n are similarly (oppositely) unitary, then*

$$\Gamma_{b,k}(k)\Gamma_{b,k}((mk+nk+k)) \geq (\leq)\Gamma_{b,k}((mk+k))\Gamma_{b,k}((nk+k)); b \geq 0. \tag{3}$$

Proof. Consider the mappings $f, g, h : [0,\infty) \to [0,\infty)$ defined by

$$f(t) = t^{mk}, \quad g(t) = t^{nk}, \quad h(t) = t^{k-1}e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}.$$

If the conditions of Theorem 2.6 hold, then the mappings f and g are synchronous (asynchronous) on $[0,\infty)$. Thus, by Chebychev integral inequality along with the choice of the functions f, g and h defined, we have

$$\int_0^\infty t^{k-1} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt \int_0^\infty t^{mk} t^{nk} t^{k-1} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt$$
$$\geq (\leq) \int_0^\infty t^{mk} t^{k-1} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt \int_0^\infty t^{nk} t^{k-1} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt.$$

This implies that

$$\int_0^\infty t^{k-1} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt \int_0^\infty t^{mk+nk+k-1} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt$$
$$\geq (\leq) \int_0^\infty t^{mk+k-1} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt \int_0^\infty t^{nk+k-1} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}dt.$$

Thus by definition of extended k-gamma function, we have

$$\Gamma_{b,k}(k)\Gamma_{b,k}((mk+nk+k)) \geq (\leq)\Gamma_{b,k}(mk+k)\Gamma_{b,k}(nk+k).$$

□

remark 2.9. *If $b = 0$, then we have the following result of classical gamma function*

$$\Gamma_b((m+n)k) \geq (\leq) \frac{kmn\Gamma_k(mk)\Gamma_k(nk)}{(m+n)}. \tag{14}$$

see [21].

Lemma 2.10. *(Hölder's Inequality see* [22]*.) If p and q are positive real numbers satisfying the condition $\frac{1}{p}+\frac{1}{q}=1$, then for integrable functions f, g: $[a, b] \to \mathbb{R}$, we have*

$$\Big| \int_a^b f(x)g(x)dx \Big| \leq \Big(\int_a^b |f|^p dx \Big)^{\frac{1}{p}} \Big(\int_a^b |g|^q dx \Big)^{\frac{1}{q}}.$$

Theorem 2.11. *Let p and q be positive real numbers satisfying the condition $\frac{1}{p}+\frac{1}{q}=1$, then prove that extended k-gamma function $\Gamma_{b,k} : (0,\infty) \to \mathbb{R}$ is log convex or $\log \Gamma_{b,k}$ is convex.*

Proof. As

$$\Gamma_{b,k}(\frac{x}{p}+\frac{y}{q}) \leq (\Gamma_{b,k}(x))^{\frac{1}{p}}(\Gamma_{b,k}(y))^{\frac{1}{q}}. \tag{15}$$

(see [16]). Let $\lambda = \frac{1}{p}$ and $(1-\lambda) = \frac{1}{q}$, then $\lambda \in (0, 1)$ and

$$\Gamma_{b,k}(\lambda x + (1-\lambda)y) \leq (\Gamma_{b,k}(x))^{\lambda}(\Gamma_{b,k}(y))^{(1-\lambda)}.$$

This implies that

$$\log(\Gamma_{b,k}(\lambda x + (1-\lambda)y)) \leq \lambda \log \Gamma_{b,k}(x) + (1-\lambda) \log \Gamma_{b,k}(y)$$

for $x, y \in (0,\infty)$, thus $\log \Gamma_{b,k}$ is convex *i.e.*, $\Gamma_{b,k}$ is log-convex. □

remark 2.12. *By Theorem 2.11, the function $\Gamma_{b,k}$ is log-convex. Also, every log-convex function is convex* [20]*, so the extended k-gamma function is convex.*

Theorem 2.13. *The function $b \to B_{b,k}(x, y)$ is logarithmically convex on $(0,\infty) \times (0,\infty)$ for each fixed $x, y > 0$.*
In particular, the following inequalities holds:

$$(i) \quad B^2_{b,k}(\frac{x_1+x_2}{k} + \frac{y_1+y_2}{k}) \leq B_{b,k}(x_1, y_1)B_{b,k}(x_2, y_2),$$

$$(ii) \quad \Big[B_{b,k}(x, y)\Big]^2 \leq B_{b,k}(x+p, y+q)B_{b,k}(x-p, y-q).$$

Proof. Let $(p, q), (m, n) \in (0,\infty)^2$, and $c, d \geq 0$ with $c+d = 1$, then we have

$$B_{b,k}(c(p, q) + d(m, n)) = B_{b,k}(cp + dm, cq + dn). \tag{16}$$

Applying the definition of extended k-beta function on the right hand side of above inequality , we get

$$B_{b,k}(c(p,q)+d(m,n))=\int_0^1 t^{\frac{cp+dm}{k}-1}(1-t)^{\frac{cq+dn}{k}-1}e^{-\frac{b^k}{kt(1-t)}}\,dt$$

$$=\int_0^1 t^{\frac{cp+dm}{k}-(c+d)}(1-t)^{\frac{cq+dn}{k}-(c+d)}e^{-\frac{b^k}{kt(1-t)}(c+d)}dt$$

$$=\int_0^1 t^{c(\frac{p}{k}-1)}t^{d(\frac{m}{k}-1)}(1-t)^{c(\frac{q}{k}-1)}(1-t)^{d(\frac{n}{k}-1)}e^{-\frac{b^k c}{kt(1-t)}}e^{-\frac{b^k d}{kt(1-t)}}\,dt$$

$$=\Bigg(\int_0^1 t^{\frac{p}{k}-1}(1-t)^{\frac{q}{k}-1}e^{-\frac{b^k}{kt(1-t)}}\Bigg)^c\Bigg(\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}e^{-\frac{b^k}{kt(1-t)}}\,dt\Bigg)^d.$$

Choose $p=\frac{1}{c}, q=\frac{1}{d}\Longrightarrow(\frac{1}{p}+\frac{1}{q}=c+d=1, p\geq 1)$. Using the Lemma 2.10 , we have

$$\leq\frac{1}{k}\Big[\int_0^1 t^{\frac{p}{k}-1}(1-t)^{\frac{q}{k}-1}e^{-\frac{b^k}{kt(1-t)}}\,dt\Big]^c\times\Big[\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}e^{-\frac{b^k}{kt(1-t)}}\,dt\Big]^d.$$

Thus, we get

$$B_{b,k}\Big[c(p,q)+d(m,n)\Big]\leq\frac{1}{k}\Big[kB_{b,k}(p,q)\Big]^c\Big[kB_{b,k}(m,n)\Big]^d$$
$$=k^{c+d-1}\Big[B_{b,k}(p,q)\Big]^c\Big[B_{b,k}(m,n)\Big]^d.$$

Here, $\lambda=c$, $(1-\lambda)=d$, then $\lambda\in(0,1)$ which shows the logarithmic convexity of β_k on $(0,\infty)^2$. □

Lemma 2.14. *The function* $b\rightarrow\frac{B_{b,k}(x-k,y-k)}{B_{b,k}(x,y)}$ *is decreasing on* $(0,\infty)$ *for fixed* $x,y>0$.
For $c=d=\frac{1}{2}$*, the above inequality reduces to*

$$B^2_{b,k}(\frac{x_1+x_2}{k}+\frac{y_1+y_2}{k})\leq B_{b,k}(x_1,y_1)B_{b,k}(x_2,y_2). \tag{17}$$

To prove (ii), Suppose that $x,y>0$ *be such that* $\min(x+a,x-a)>0$*, then* $x_1=x+p$*,* $x_2=x-p$ *and* $y_1=y+q$*,* $y_2=y-q$ *in equation (17) gives*

$$\Big[B_{b,k}(x,y)\Big]^2\leq B_{b,k}(x+p,y+q)B_{b,k}(x-p,y-q). \tag{18}$$

Proof. The log-convexity of $B_{b,k}(x,y)$ is equivalent to

$$\frac{d}{db}\left(\frac{\frac{d}{db}B_{b,k}(x,y)}{B_{b,k}(x,y)}\right)\geq 0. \tag{19}$$

Now one can get the following identity of extended k-beta function

$$\frac{d^n}{db^n}B_{b,k}(x,y)=(-1)^n(b)^{nk-n}B_{b,k}(x-nk,y-nk); n=0,1,\cdots,k>0,b>0. \tag{20}$$

Thus (19) reduces to

$$\frac{d}{db}\left(\frac{B_{b,k}(x-k,y-k)}{B_{b,k}(x,y)}\right)\leq 0. \tag{21}$$

Hence the result follows. □

Lemma 2.15. *Let f and g be two integrable functions on $[a,b]$ and $h : [a,b]\to[0,\infty)$ is such that $\int_a^b h(x)dx>0$. If $m\leq f(t)\leq M$ and $l\leq g(t)\leq L$, for each $t\in[a,b]$, where m,M,l,L are given real constant. Then*

$$\Big|\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)g(x)h(x)dx-\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)h(x)dx\Big|$$
$$\leq\frac{1}{4}(M-m)(N-n)$$

and the constant $\frac{1}{4}$ is best possible see [10].

Theorem 2.16. *Let m,n,p,q and k be positive real numbers and r, $s>-k$ then, we have*

$$\Big|B_{b,k}(r+k,s+k)\beta_k(m+p+r+k,n+q+s+k)-B_{b,k}(m+r+k,n+s+k)B_{b,k}(p+r+k,q+s+k)\Big|$$

$$\leq\frac{1}{4k}\frac{m^{\frac{m}{k}}n^{\frac{n}{k}}}{(m+n)^{\frac{m+n}{k}}}\frac{p^{\frac{p}{k}}q^{\frac{q}{k}}}{(p+q)^{\frac{p+q}{k}}}B^2_{b,k}(r+k,s+k). \tag{39}$$

Proof. Consider the functions defined by

$$f_{m,n}(x)=x^{\frac{m}{k}}(1-x)^{\frac{n}{k}}=f(x)\quad,\qquad f_{p,q}(x)=x^{\frac{p}{k}}(1-x)^{\frac{q}{k}}=g(x),$$
$$f_{r,s}(x)=x^{\frac{r}{k}}(1-x)^{\frac{s}{k}}e^{-\frac{b^k}{kx(1-x)}}=h(x)\quad,\quad x\in[0,1],\ k>0.$$

For the application of Grüss' inequality, we have to find the minima and maxima of $f_{a,b}(x)$, $(a,b,k>0)$. Thus

$$\frac{d}{dx}f_{a,b}(x)=\frac{1}{k}\,x^{\frac{a}{k}-1}(1-x)^{\frac{b}{k}-1}[a-(a+b)x].$$

Here, we see that the solution of $f'_{a,b}(x)=0$ in the interval $(0,1)$ is $x_0=\frac{a}{a+b}$. Also, $f'_{a,b}(x)>0$ on $(0,x_0)$ and $f'_{a,b}(x)<0$ on $(x_0,1)$. We conclude that x_0 is the maximum point in the interval $(0,1)$ and consequently

$$m_{a,b}=\inf{}_{\cdot x\in[0,1]}f_{a,b}(x)=0=m(say)$$

$$and\quad M_{a,b}=\sup{}_{\cdot x\in[0,1]}f_{a,b}(x)=f_{a,b}(\frac{a}{a+b})=\frac{a^{\frac{a}{k}}b^{\frac{b}{k}}}{(a+b)^{\frac{a+b}{k}}}=M(say).$$

Hence, by Grüss' inequality 2.15, we have

$$\Rightarrow\Big|\int_0^1 x^{\frac{r}{k}}(1-x)^{\frac{s}{k}}e^{-\frac{b^k}{kx(1-x)}}dx.\int_0^1 x^{\frac{p}{k}}(1-x)^{\frac{q}{k}}x^{\frac{r}{k}}(1-x)^{\frac{s}{k}}e^{-\frac{b^k}{kx(1-x)}}dx$$
$$-\int_0^1 x^{\frac{m}{k}}(1-x)^{\frac{n}{k}}x^{\frac{r}{k}}(1-x)^{\frac{s}{k}}e^{-\frac{b^k}{kx(1-x)}}dx\times\int_0^1 x^{\frac{p}{k}}(1-x)^{\frac{q}{k}}x^{\frac{r}{k}}(1-x)^{\frac{s}{k}}e^{-\frac{b^k}{kx(1-x)}}dx\Big|$$
$$\leq\frac{1}{4k}\frac{p^{\frac{p}{k}}q^{\frac{q}{k}}}{(p+q)^{\frac{p+q}{k}}}\frac{r^{\frac{r}{k}}s^{\frac{s}{k}}}{(r+s)^{\frac{r+s}{k}}}\Big[\int_0^1 x^{\frac{r}{k}}(1-x)^{\frac{s}{k}}e^{-\frac{b^k}{kx(1-x)}}dx\Big]^2.$$

Rearranging the terms on left hand side and using the relation (6) with simple algebraic computation, we reach the required proof. □

Theorem 2.17. *Let p, q, and k be positive real numbers and r, $s > -k$, then*

$$\Big|B_{b,k}(r+k, s+k)B_{b,k}(p+r+k, q+s+k) - B_{b,k}(p+r+k, s+k)B_{b,k}(r+k, q+s+k)\Big|$$

$$\leq \frac{1}{4k}B^2_{b,k}(r+k, s+k).$$

Proof. Using Lemma 2.15 by considering the choice of functions defined by

$$x^{\frac{p}{k}} = f(x),\ (1-x)^{\frac{q}{k}} = g(x),\quad f_{r,s}(x) = x^{\frac{r}{k}}(1-x)^{\frac{s}{k}} = h(x),\ x \in [0,1],\ k > 0,$$

Clearly, $M = L = 1$ and $m = l = 0$. Thus we have the following inequality

$$\Big|B_{b,k}(r+k, s+k)B_{b,k}(p+r+k, q+s+k) - B_{b,k}(p+r+k, s+k)B_{b,k}(r+k, q+s+k)\Big|$$
$$\leq \frac{1}{4k}B^2_{b,k}(r+k, s+k)$$

□

Lemma 2.18. *(see [19] p. 295-310) Let f and g be two integrable functions on $[a,b]$ and $h : [a,b] \to [0,\infty)$ is such that $\int_a^b h(x)dx > 0$. If $m \leq f(t) \leq M$ and $l \leq g(t) \leq L$, for each $t \in [a,b]$, where m, M, l, L are given real constant. Then*

$$\Big|D(f,g;h)\Big| \leq D(f,f;h)^{\frac{1}{2}}D(g,g;h)^{\frac{1}{2}} \leq \frac{1}{4}(M-m)(N-n)\Big[\int_a^b h(t)dt\Big]^2$$

where

$$D(f,g;h) = \int_a^b h(t)dt\int_a^b h(t)f(t)g(t)dt - \int_a^b h(t)f(t)dt\int_a^b h(t)g(t)dt.$$

Theorem 2.19. *Let $b_1, b_2, x, y > 0$. Then the following inequality holds:*

$$\Big|B_{(b_1^k+b_2^k)^{\frac{1}{k}},k}(x+y+k, x+y+k) - B_{b_1,k}(x+k, y+k)B_{b_2,k}(x+k, y+k)\Big|$$
$$\leq \Big[B_{2b_1,k}(2x+k, 2x+k) - B_{b_1,k}(x+k, x+k)^2\Big]^{\frac{1}{2}}$$
$$\times \Big[B_{2b_1,k}(2y+k, 2x+k) - B_{b_1,k}(y+k, y+k)^2\Big]^{\frac{1}{2}}$$

$$\leq \frac{\exp\Big(-4(\frac{b_1^k+b_2^k}{k})\Big)}{4^{x+y+1}k} \tag{22}$$

Proof. Consider the function

$$f(t) = x^{\frac{x}{k}}(1-t)^{\frac{x}{k}}\exp\Big(-\frac{b_1^k}{kt(1-t)}\Big)$$
$$g(t) = x^{\frac{y}{k}}(1-t)^{\frac{y}{k}}\exp\Big(-\frac{b_1^k}{kt(1-t)}\Big),$$

for $t \in [0,1]$ and $x, y, b_1, b_2 > 0$. It is clear that $f(0) = f(1) = 0$ and $g(0) = g(1) = 0$. Now for $t \in (0,1)$, we have

$$f'(t) = \frac{1}{k} f(t)(1-2t)\Big(\frac{kxt(1-t) + b_1^k}{kt^2(1-t)^2}\Big).$$

Since $f(t) > 0$ and $kxt(1-t) + b_1^k > 0$ on $t \in (0,1)$, $f'(t) > 0$ for $t > \frac{1}{2}$ and $f'(t) < 0$ for $t < \frac{1}{2}$. This implies

$$M = \frac{\exp\Big(-4\frac{b_1^k}{k}\Big)}{4^x}.$$

Similarly,

$$L = \frac{\exp\Big(-4\frac{b_2^k}{k}\Big)}{4^y}.$$

Now using f, g as defined above and taking $h(t) = 1$ for all $t \in [0,1]$ in lemma 2.18 yields (22). □

3. CONCLUSION

In this paper, we conclude that if f and g asynchronous then all the obtained inequalities will be reverse. Moreover, the obtained inequalities are the generalization of recently proved result of Mondal [18] and the extended form of some of the result of Rehman *et al.* [21]. It is clear that if letting $k \to 1$, then our obtained results will reduce to the results of extended beta function see [18]. Similarly, if letting $b = 0$, then we get some results of k-beta function earlier proved in [21].

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